Final take-home: Topics in Modern Mathematical Physics II

Finish either Problem 1 or 2, and I will grade accordingly. DUE 1/21 5:00pm.

You can send your solutions to me by email, either directly (b-fang@bicmr.pku.edu.cn) or in Courseworks. I will reply within 12 hours to confirm that I have received your submission.

Throughout, work on $\overline{\mathcal{M}}_{g,n}$, assume stability $2g-2+n>0$, and use the convention

$$\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}.$$

Problem 1 (String and dilaton equations)

Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map forgetting the last marking and stabilizing.

1. Prove the **string equation**

$$\langle \tau_0\tau_{d_1}\cdots\tau_{d_n}\rangle_g = \sum_{i=1}^{n}\langle \tau_{d_1}\cdots\tau_{d_i-1}\cdots\tau_{d_n}\rangle_g \qquad (\text{with } \langle\cdots\tau_{-1}\cdots\rangle_g := 0).$$

   and show that

$$\langle \tau_0\tau_2\rangle_1 = \langle \tau_1\rangle_1.$$

2. Prove the **dilaton equation**

$$\langle \tau_1\tau_{d_1}\cdots\tau_{d_n}\rangle_g = (2g-2+n)\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_g.$$

   and show that

$$\langle \tau_1\tau_1\rangle_1 = \langle \tau_1\rangle_1.$$

Problem 2 (Ribbon graphs / arc complex for $\mathcal{M}_{1,2}$)

Consider the arc systems on a genus-1 surface with two punctures and dual ribbon graphs. The top cells of the corresponding ribbon-graph decomposition of the moduli have trivalent ribbon graphs with $g = 1, n = 2$.

1. What is the number of vertices, edges, and faces of such a trivalent ribbon graph?
2. Classify all such ribbon graphs.
3. Using the ribbon-graph Laplace transform identity stated as Equation 4.4 (p732) in *Geometry of Algebraic Curves II*, Ch. XX for the case $g = 1$ and $n = 2$, extract

$$\int_{\overline{\mathcal{M}}_{1,2}} \psi_1 \psi_2.$$