

Clutching maps and automorphisms of dual graphs

Separating and nonseparating boundary strata

We work over $\mathbb{C}$. Write $\mathcal{M}_{g,n}$ for the moduli stack of smooth pointed curves and $\overline{\mathcal{M}}_{g,n}$ for its stable compactification, with $2g - 2 + n > 0$. Markings are labeled. An *attaching map*, also called a *clutching map*, glues two marked points to make a node.

1. What does “irr” mean?

The subscript irr abbreviates **irreducible**. The divisor Δ_{irr} , also often denoted Δ_0 , is the closure of the locus of irreducible curves with exactly one node. Its attaching map is

$$\xi_{\text{irr}} : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad (C, p_1, \dots, p_n, p, q) \longmapsto C/(p \sim q),$$

with the first n markings retained. For smooth C , gluing p to q produces an irreducible curve of arithmetic genus g .

A node is *separating* if normalizing at that node disconnects the curve, and *nonseparating* if it leaves the curve connected. A curve with exactly one node has a nonseparating node precisely when it is irreducible. The closure Δ_{irr} can also contain reducible curves with more nodes.

2. Two simple failures of injectivity

Separating: two elliptic components. Consider

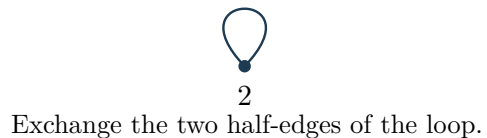
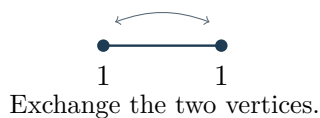
$$\xi_1 : \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1} \longrightarrow \overline{\mathcal{M}}_2, \quad ((E_1, p_1), (E_2, p_2)) \longmapsto E_1 \cup_{p_1 \sim p_2} E_2.$$

Interchanging the two factors gives the same glued curve up to isomorphism. If (E_1, p_1) and (E_2, p_2) are nonisomorphic, the two ordered pairs are distinct points of the source. Thus the induced map on coarse moduli spaces is generically $2 : 1$ onto its image, and the clutching map is not an embedding.

Nonseparating: gluing two points of a genus-2 curve. Consider

$$\xi_{\text{irr}} : \overline{\mathcal{M}}_{2,2} \longrightarrow \overline{\mathcal{M}}_3, \quad (C, p, q) \longmapsto C/(p \sim q).$$

The triples (C, p, q) and (C, q, p) have the same image. For a general genus-2 curve and general distinct p, q , no automorphism interchanges p and q : the only nonidentity automorphism is the hyperelliptic involution, and we choose q outside its orbit of p . The two triples are therefore nonisomorphic. This map is again generically $2 : 1$ on coarse moduli spaces.



Vertex labels record the genera of the normalized components. In both examples the graph automorphism group is S_2 .

3. The general statement for a fixed dual graph

Let Γ be a stable graph. Its vertices represent normalized irreducible components, its edges represent nodes, and its labeled legs represent the original markings. A vertex v carries its genus $g(v)$; let $n(v)$ count all incident legs and half-edges. Set

$$\mathcal{M}_\Gamma = \prod_{v \in V(\Gamma)} \mathcal{M}_{g(v), n(v)}.$$

The markings on each factor are individually labeled by the corresponding legs and half-edges. Gluing the paired half-edges defines

$$\xi_\Gamma : \mathcal{M}_\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad g = \sum_v g(v) + b_1(\Gamma).$$

Let $\mathcal{S}_\Gamma$ be the locally closed stratum of curves whose dual graph is *exactly* Γ . Then

$$\mathcal{S}_\Gamma \simeq [\mathcal{M}_\Gamma / \text{Aut}(\Gamma)].$$

Here $\text{Aut}(\Gamma)$ preserves the incidence data, the genus decorations, and every labeled external leg. It can permute vertices and half-edges, including the two half-edges of a loop. Square brackets denote a *stack quotient*, retaining stabilizers.

Why? Normalize a curve in $\mathcal{S}_\Gamma$ at all its nodes. An identification of its dual graph with Γ labels its components and all the preimages of nodes, producing an object of $\mathcal{M}_\Gamma$. Changing that identification acts by $\text{Aut}(\Gamma)$. Conversely, gluing these labeled normalization data recovers the curve, and isomorphisms lift uniquely to the normalization. This gives the quotient description, also for families after local labeling.

For the open one-node strata in the two examples,

$$\Delta_1^\circ \simeq [(\mathcal{M}_{1,1} \times \mathcal{M}_{1,1})/S_2], \quad \Delta_{\text{irr}}^\circ \simeq [\mathcal{M}_{2,2}/S_2].$$

4. Graph symmetries versus automorphisms of a curve

A graph symmetry need not be realized by an automorphism of a particular curve. For a pointed nodal curve C with normalized components C_v , let P_v denote the individually labeled special points on C_v . There is an exact sequence

$$1 \longrightarrow \prod_v \text{Aut}(C_v; P_v) \longrightarrow \text{Aut}(C) \longrightarrow H_C \longrightarrow 1, \quad H_C \subseteq \text{Aut}(\Gamma_C),$$

where H_C is the image of the action on the dual graph. Each group in the product fixes every special point separately. These component automorphisms are already present in the factors of $\mathcal{M}_\Gamma$.

In the separating example with $E_1 \not\cong E_2$, no automorphism of the glued curve exchanges its components, although the two orderings still have the same image. If the two pointed elliptic curves are identical, exchanging the components *is* an extra automorphism; for general (E, p) ,

$$\text{Aut}(E \cup_{p \sim p} E) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2.$$

The two factors come from the elliptic involutions fixing p ; S_2 exchanges them.

Smallest stack example. Glue 0 and ∞ on $(\mathbb{P}^1; 1, 0, \infty)$. The resulting one-pointed nodal genus-1 curve has the involution $z \mapsto 1/z$. Thus

$$\mathcal{M}_{0,3} = \text{pt} \longrightarrow \Delta_{\text{irr}}^\circ \subset \overline{\mathcal{M}}_{1,1} \quad \text{is} \quad \text{pt} \longrightarrow B(\mathbb{Z}/2\mathbb{Z}).$$

There is one coarse point on each side, but this is not a stack embedding: the target has an extra stabilizer. More generally, $\mathcal{M}_\Gamma \rightarrow \mathcal{S}_\Gamma$ has stack degree $|\text{Aut}(\Gamma)|$, while a coarse fiber can have fewer points.

5. What changes on the closure?

Now allow the vertex curves themselves to degenerate, and set

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v), n(v)}.$$

The proper clutching map factors as

$$\overline{\mathcal{M}}_\Gamma \longrightarrow [\overline{\mathcal{M}}_\Gamma / \text{Aut}(\Gamma)] \xrightarrow{\nu_\Gamma} \overline{\mathcal{S}}_\Gamma \hookrightarrow \overline{\mathcal{M}}_{g,n}.$$

The map ν_Γ is the **normalization** of the reduced stratum closure. It is an isomorphism over $\mathcal{S}_\Gamma$, but need not be an isomorphism over deeper boundary strata. In particular, one must not in general identify the entire boundary closure with the quotient of the compactified source.

The reason is concrete. A degenerate image curve may have extra nodes, and the clutching source remembers *which nodes were created by the original edges of Γ* . Different choices need not be related by $\text{Aut}(\Gamma)$ or by an automorphism of the image curve.

Example: two possible nodes to unglue. Let E be a smooth elliptic curve and choose four general distinct points p_1, q_1, p_2, q_2 . Form

$$D = E / (p_1 \sim q_1, p_2 \sim q_2).$$

This is an irreducible stable curve of arithmetic genus 3 with two nodes. Choose the points generally enough that D has no nonidentity automorphism. Under

$$\xi_{\text{irr}} : \overline{\mathcal{M}}_{2,2} \longrightarrow \overline{\mathcal{M}}_3,$$

it has preimages obtained by normalizing either node:

$$\begin{aligned} x_1 &= (E / (p_2 \sim q_2), p_1, q_1), \\ x_2 &= (E / (p_1 \sim q_1), p_2, q_2). \end{aligned}$$

Both underlying source curves have arithmetic genus 2. Swapping the last two markings of either x_i gives two further preimages. Hence there are four coarse preimages, forming *two* S_2 -orbits. The classes of x_1 and x_2 remain distinct in $[\overline{\mathcal{M}}_{2,2}/S_2]$, yet they map to the same D .

If t_1, t_2 are the two node-smoothing parameters, then on a local deformation space of D the divisor Δ_{irr} has equation

$$t_1 t_2 = 0$$

(with the other parameters unrestricted). Its normalization separates these two local branches. Thus Δ_{irr} is globally irreducible, but can have several local branches along the deeper boundary.

For the classroom. With the dual graph fixed, the stratum is the product of the moduli of the normalized components, modulo graph automorphisms. On the closure, one must additionally remember which nodes came from the prescribed gluing. Component automorphisms are already present in the source; graph symmetries account for the extra identifications on the open stratum.

Reference. D. Edidin and D. Fulghesu, *Moduli stacks of curves with a fixed dual graph*, [arXiv:1007.2130](https://arxiv.org/abs/1007.2130), Proposition 4.4 and Theorem 4.5 (the unmarked case; the labeled-point version has the same normalization-and-gluing argument).